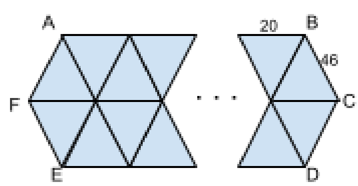
Geometry Test

1. (3) A cone-shaped bucket is filled to 12.5% of its volume. If the height of the bucket is 6, what is the height of the water in the bucket?
Answer: 3
Solution: *Since the shapes of the conic bucket and the water are similar, their volumes are proportional in the form* $\frac{m^3}{n^3}$*, while the ratio of the heights are proportional in the form* $\frac{m}{n}$*. From here, you can plug in some values and solve for m ->* $\frac{12.5}{100} = \frac{m^3}{(6)^3}$ *->* $\frac{1}{8} = \frac{m^3}{216}$ *-> 216 =* $8m^3$ *->* $27 = m^3$ *-> m = 3.*

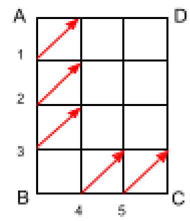
2. (3) A right triangle has two sides of length 5 and 12. What is the shortest possible length of the remaining side of the triangle? This length can be expressed in the form $\sqrt{a}$; what is *a*?
Answer: 119
Solution: *In a right triangle, the hypotenuse has to be the longest side. This means that either the side of the length 12 or the unknown length has to be the hypotenuse. If the unknown is the hypotenuse, then it is obviously 13 =* $\sqrt{169} = 5^2 + 12^2$*. If 12 is the length of the hypotenuse, then the unknown is* $\sqrt{144 - 25} = \sqrt{119}$*.*

3. (3) 42 congruent, isosceles triangles with congruent sides of length 46 and bases of length 20 are placed in the pattern forming hexagon ABCDEF, as shown. What is the perimeter of hexagon ABCDEF?



Answer: 584
Solution: *Since there trapezoids ABCF and DEFC are congruent, they each contain 42 / 2 = 21 of the triangles. For ABCF, 10 triangles have bases on AB while 11 have bases on FC. Therefore, AB = DE = 10 x 20 = 200 . BC = FA = CD = EF = 46 . 2(200) + 4(46) = 584.*

4. (4) In Knospe-Ball, a ball can be launched from points 1, 2, 3, 4, or 5 in the directions shown. When a ball hits a side of rectangle ABCD it bounces at a 90° angle back into the playing field. The path of the ball ends when it hits a corner point A, B, C, or D. Each of the gridded squares has a perimeter of 12. What is the length of the longest possible path for a ball launched from a starting point? This length can be expressed in the form $a\sqrt{b}$; what is $a + b$?.



Answer: 29

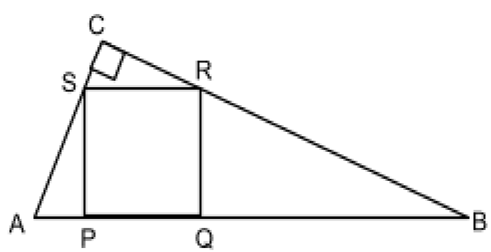
Solution: *Obviously, the sidelength of each grid square is 12/4 = 3. Therefore, the path through each grid square is* $3\sqrt{2}$*. For the purposes of this explanation, compass directions will be used. For Path 1, the ball goes NE for one square, then SE for 2 squares, SW 2, NW 1, and then NE 3 before hitting D and stopping. For Path 2, the ball goes NE 2, SE 1, and then SW 3 into B. For Path 3, the ball goes NE 3 into D. For Path 4, the ball goes NE 2, NW 2, SW 1, and then SE 3 into C. For Path 5, the ball goes NE 1, and then NW 3 into A. Path 1 is the longest path, going through 9 squares, so the length of the path is* $27\sqrt{2}$*. 27 + 2 = 29.*

5. (4) Triangle ABC with vertices A(-2,0), B(1,4), and C(-3,2) is reflected over the line y = x to form triangle A'B'C'. What is the length of CC'? This length can be expressed in the form $\sqrt{a}$; what is *a*?

Answer: 50

Solution: *Points reflected over the y = x line just have their x and y coordinates flipped. Therefore, C' is at (2,-3). Using the distance formula,* $CC' = \sqrt{(-3-2)^2 + (2+3)^2} = \sqrt{50}$

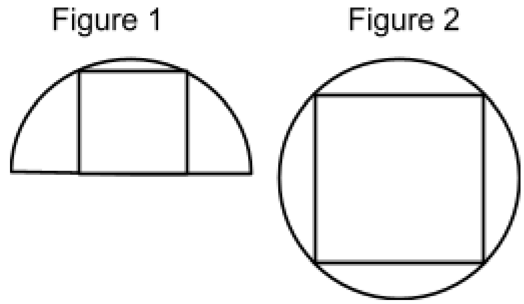
6. (5) Square PQRS is inscribed in right triangle ABC, as shown. If AP = 24 and QB = 60, what is the area of square PQRS?



Answer: 1440

Solution: *Triangles APS and RQB are similar to triangle ACB. To prove this, we know that <C , <APS , and <BQR are all 90º. <A and <B must sum to 90º, but <A + <ASP = 90º = <B + <QRB . From this conclusion, let's say PQ = x. The ratio of x to AP is the same as the ratio of QB to x ->* $\frac{x}{24} = \frac{60}{x}$*->* $x^2 = 1440$*. Since the area of square PQRS is equal to* x^2*, 1440 is our answer.*

7. (5) The area of the semicircle in Figure 1 is half the area of the circle in Figure 2. A square is inscribed in the semicircle while another square is inscribed in the circle. If the area of the larger square is 160 square units, what is the area of the square inscribed in the semicircle?



Answer: 64

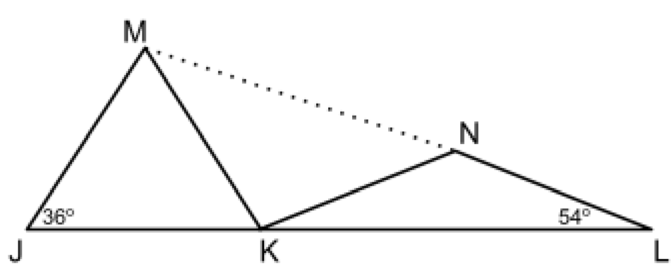
Solution: *The side length of the larger square is* $\sqrt{160} = 4\sqrt{10}$*. This means that the diagonal, which is the circle's diameter, has a length of* $\sqrt{320} = 8\sqrt{5}$*. The radius of the circle is therefore* $4\sqrt{5}$*. On the semicircle, this radius goes from the midpoint of one side of the square to the opposite corner on that square. Since the sides of the resulting right triangle are in the ratio 2:1, we can treat a side of the square as x and say that* $(4\sqrt{5})^2 = x^2 + (0.5x)^2$ *-> 80 =* $\frac{5}{4}x^2$*.* x^2 *= 64, which is the area of the square.*

8. (5) An isosceles trapezoid has legs of length 15, two diagonals of length 20, and the longer base has a length of 25. What is the trapezoid's area? This can be expressed in the form $a + b\sqrt{c}$; what is a + b + c?

Answer: 153

Solution: *Since the leg, diagonal, and base make a right triangle ($15^2 + 20^2 = 25^2$), the area of the triangle is (.5)(15)(20) = 150. This area is also equal to the base times the height of the trapezoid -> h =* $\frac{150}{25} = 6$*. In the right triangle with one of the legs as the hypotenuse and the height as one side, we can find the half of the difference between the longer base and the shorter base -> $15^2 - 6^2 = x^2$ -> $225 - 36 = x^2$ -> x =* $3\sqrt{21}$*. Therefore, the length of the shorter base is* $25 - 6\sqrt{21}$*. The area of the trapezoid is therefore*
$(.5)(25 + 25 - 6\sqrt{21})(6) = 150 - 18\sqrt{21}$*. 150 - 18 + 21 = 153.*

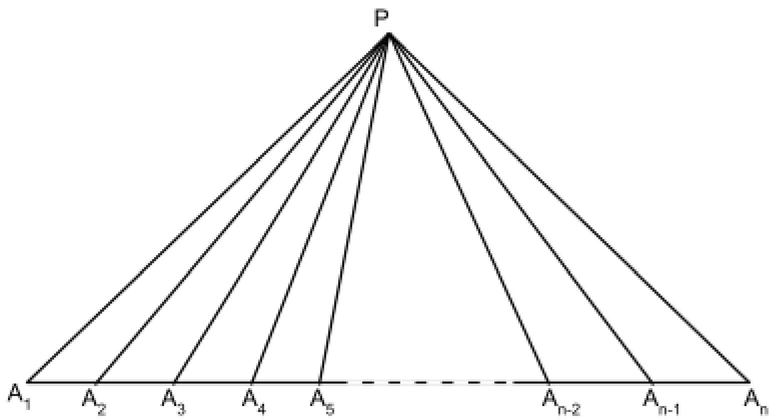
9. (6) Coplanar points J, K, L, M, and N are arranged such that J, K, and L are collinear with K between J and L. Triangles JKM and KNL are both isosceles and have equal areas. JM and MK are the congruent legs of triangle JKM while KN and NL are the congruent legs of triangle KNL. <MJK = 36° while <NLK = 54°. Points M and N are on the same side of JL. What is the degree measure of <KMN?



Answer: 45

Solution: *Since J, K, and L are collinear, we can find the measure of <MKN = 180 - 36 - 54 = 90^{o}. Next, we must notice that <KNL = 72^{o} which is twice that of <J. If we drop a perpendicular from N to the midpoint of KL (which we'll call point P), we bisect <KNL. If we drop a perpendicular from M to the midpoint of JK (which we'll call point Q), we bisect <JKM. Triangles KQM and NPK are congruent since their angles are the same and they have the same area. Therefore, MK = KN, so triangle KMN is an isosceles right triangle. <KMN must then equal 45^{o}.*

10. (6) The figure below contains a total of 2016 triangles and *n* points labeled as vertices on the horizontal base. What is the value of *n*?



Answer: 64

Solution: *There has to be a series to count all of the triangles. We should start with n = 2, which gives 1 triangle. If n = 3, there are 3 triangles (2 small and 1 large). If n = 4, there are 6 triangles (3 small, 2 medium, and 1 large). These are the triangle numbers, which can be written as $\frac{n(n+1)}{2}$. Since the first triangle number goes with n = 2, we need to plug in (n - 1) for n -> a_n = $\frac{n(n-1)}{2}$. Now we need to solve for n -> 2016 = $\frac{n^2-n}{2}$-> 0 = n^2 - n - 4032. The easiest way to factor this is to realize that the factors of 4032 that we want differ by only 1. $\sqrt{4032} \approx 63$. Since we need a 2 in the units digit, the other factor must be 64; this is correct if you take the time to check. Therefore, 0 = (n - 64)(n + 63) -> n = 64.*